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Spin Phase

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The existence of a complementary quantity in the Weyl sense for the third component of spin is discussed. This is called the spin phase and the possibilities of measuring this quantity are considered. Connections between the spin phase and the indeterminacy of the direction for the spin are shown.

1. INTRODUCTION

The question of the phase of a spin seems to be strange. Sometimes one thinks about the orbital angular momentum more or less via the classical picture. It is a picture of a vector in space. One of the angles determining the direction is simply complementary to the third component of the orbital angular momentum and is called the phase of the angular momentum. The counterpart of this notion for spin is more sophisticated. Many textbooks on quantum mechanics argue that the spin has such nonclassical properties that it cannot be represented in any case in a classical way. Further, the operator which should describe the spin phase should possess a continuous spectrum. At first glance such a property is impossible for the finite-dimensional case.

I will show that the construction of the spin phase can be done in a way analogous to the harmonic oscillator case. This description requires the extension of the notion of observable in quantum mechanics in the spirit of Davies (1976), Holevo (1982), and Ludwig (1983) (so-called generalized observables, semispectral measures, semiobservables, effects). As a result, I obtain the semispectral measure complementary in the Weyl sense to the third component of spin.

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In considering an experiment which enables the measurement of such a quantity, I examine the recent experiment on spin superposition by Summhammer *et al.* (1983). Without such an analysis the construction of the spin phase has only a purely mathematical sense and its physical sense is highly incomplete.

Furthermore, I discuss the description of the direction of spin within the extended quantum formalism with semiobservables. I obtain the connections between the phase and the direction of spin.

In addition, the entropic formulation of the "phase-third spin component" uncertainty relation is given.

2. FORMAL DESCRIPTION OF SPIN PHASE

Let $\mathscr{H} = \mathbb{C}^{2J+1}$ be the Hilbert space for a spin system, where $J = 1/2$, $1, 3/2, \ldots$, and $\mathbb{C}$ denotes the complex numbers. The $\mathscr{J}_i$ are spin operators $i = 1, 2, 3$ with the commutation relation

$$[\mathscr{J}_i, \mathscr{J}_j] = i\varepsilon_{ijk}\mathscr{J}_k \tag{1}$$

and $\mathscr{J}_3|m\rangle = m|m\rangle$, $m = -J, \ldots, J$. For operators $\mathscr{J}_\pm = \mathscr{J}_1 \pm i\mathscr{J}_2$ we can use the polar decomposition theorem (Reed and Simon, 1973)

$$\mathscr{J}_+ = P(\mathscr{J}_-\mathscr{J}_+)^{1/2}; \qquad \mathscr{J}_- = (\mathscr{J}_-\mathscr{J}_+)^{1/2}P^* \tag{2}$$

where P is a partial isometry with the following properties:

$$\begin{aligned} P|m\rangle &= |m+1\rangle, \qquad P|J\rangle = 0 \\ P^*|m\rangle &= |m-1\rangle, \qquad P^*|-J\rangle = 0 \end{aligned} \tag{3}$$

and, moreover,

$$[P, \mathscr{J}_3] = P \tag{4}$$

P and P^* act between the following subspaces:

$$\begin{aligned} P\colon &\quad [|+J\rangle]^\perp \to [|-J\rangle]^\perp \\ P^*\colon &\quad [|-J\rangle]^\perp \to [|+J\rangle]^\perp \end{aligned}$$

This means that

$$\begin{aligned} P^*P &= I - |J\rangle\langle J| \\ PP^* &= I - |-J\rangle\langle -J| \end{aligned} \tag{5}$$

where I is the identity operator. Using a very general theorem coming from Sz.-Nagy (1953) and Foiaç (1957), we have the spectral decomposition

$$P = \int_0^{2\pi} e^{i\varphi}\, M(d\varphi) \tag{6}$$

where $M(\cdot)$ is a semispectral measure. M is a spectral measure without the idempotency condition, i.e., $M^2 = M$. For M we can adopt here the probabilistic Born interpretation of quantum mechanics and treat M as a generalized observable. We will call M the spin phase. Simple calculations with (4) give the Weyl form of the commutation relation for $\mathcal{J}_3$ and M,

$$e^{i-i\alpha\mathcal{J}_3} M(\Delta)\, e^{i\alpha\mathcal{J}_3} = M(\Delta - \alpha) \tag{7}$$

modulo 2π for arbitrary $\Delta \in \mathcal{B}(0, 2\pi)$, the Borel σ-algebra of $(0, 2\pi)$.

We can obtain the form of M in the spectral representation of M. To build this representation, we take the Hilbert space $L^2(0, 2\pi)$ with the scalar product

$$(\eta, \xi) = \int_0^{2\pi} \overline{\eta^{(\varphi)}} \xi(\varphi)\, d\varphi/2\pi \tag{8}$$

where $\eta, \xi \in L^2(0, 2\pi)$. In $L^2(0, 2\pi)$ we choose the following subspace:

$$\tilde{\mathcal{H}} = \left\{ \psi(\varphi) = \sum_{m=-J}^{J} e^{-im\varphi} c_m,\ c_m = \langle m \mid \psi \rangle,\ \psi \in \mathcal{H} \right\} \subset L^2(0, 2\pi)$$

In $\tilde{\mathcal{H}}$ the P and P^* have the form

$$\begin{aligned} (P\psi)(\varphi) &= e^{i\varphi}[\psi(\varphi) - c_J\, e^{-iJ\varphi}] \\ (P^*\psi)(\varphi) &= e^{-i\varphi}[\psi(\varphi) - c_{-J}\, e^{iJ\varphi}] \end{aligned} \tag{9}$$

In turn, M has in $\tilde{\mathcal{H}}$ the following shape (Grabowski, to appear):

$$(\eta, M(\Delta)\xi) = \int_\Delta \overline{\eta(\varphi)} \xi(\varphi)\, d\varphi/2\pi \tag{10}$$

for $\Delta \in \mathcal{B}(0, 2\pi)$ modulo 2π. In spite of (10), the $M(\cdot)$ cannot be represented as a characteristic function of Δ (i.e., the idempotency condition does not really hold). It is easily visible for the case $J = 1/2$. The space $\tilde{\mathcal{H}}$ consists of linear combinations $\alpha \cos \varphi/2 + \mathcal{B} \sin \varphi/2$. The idempotency condition demands $(\chi_\Delta \psi)(\varphi) \in \tilde{\mathcal{H}}$, where χ_Δ is a characteristic function of Δ. We cannot represent a function which is different from zero only on Δ as a linear combination of sine and cosine. Thus $\chi_\Delta \psi \notin \tilde{\mathcal{H}}$. For the same reasons the following property holds:

$$(\psi, M(\Delta)\psi) < 1 \tag{11}$$

for all $\psi \in \tilde{\mathcal{H}}$ and any subset $\Delta \in \mathcal{B}(0, 2\pi)$, $\Delta \neq (0, 2\pi)$, with nonzero Lebesgue measure. The inequality (11) means that the localization of the phase is impossible.

We can also write the M in $\mathbb{C}^{2J+1}$. Starting from (10) and using decompositions

$$\begin{aligned} \eta(\varphi)-\sum_m a_m e^{im\varphi}; \qquad a_m=\langle m\mid\eta\rangle \\ \xi(\varphi)=\sum_{m'} b_{m'} e^{im'\varphi}; \qquad b_m=\langle m\mid\xi\rangle \end{aligned} \tag{12}$$

we obtain

$$M(\Delta)=\int_\Delta \sum_{m,m'} e^{i(m'-m)\varphi}|m\rangle\langle m'|\, d\varphi/2\pi \tag{13}$$

For further purposes we introduce the vector $|\omega(\varphi)\rangle=\sum_{m=-J}^{J} e^{im\varphi}|m\rangle$. Now we can write

$$M(\Delta)=\int_\Delta |\omega(\varphi)\rangle\langle\omega(\varphi)|\, d\varphi|2\pi=\int_\Delta P(\varphi)\, d\varphi/2\pi \tag{14}$$

The form (13) has been obtained by Holevo (1983) in quantum decision theory.

3. THE POSSIBILITY OF MEASUREMENT OF SPIN PHASE

To this point we have only a purely mathematical construction. This construction shows that the mathematical formalism of quantum mechanics and the probabilistic interpretation are large enough to encompass it. The semispectral measure M will attain the status of an observable in quantum mechanics when we provide a concrete experimental scheme of measurement of the parameter φ. The question is, what kind of experimental situation could be described by means of (14)?

Several years ago Holevo (1983) treated an expression like (13) as a statistical inference problem. It was only a mathematical prescription of how to determine the parameter φ on a certain class of states in an optimal way with respect to a chosen function. No concrete proposal of any experimental setup was given. At the same time, there appeared the possibility of preparation of a large set of states for spin 1/2 by means of neutron interferometry. These states can also be detected. Especially interesting in connection with the question of the phase of spin are the observations of fermion spin superposition by neutron interferometry performed by Summhammer *et al.* (1983). I believe that this type of experiment could in principle serve as a measurement for the spin phase.

To discuss this, we must first recall the idea of the spin superposition experiment (Figure 1). A polarized incident neutron beam is split coherently and the spin in one of the beams is inverted. At the third crystal plate, the

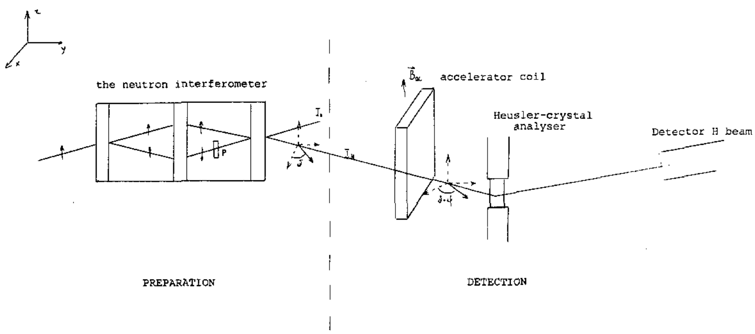

Fig. 1

two wave packets with opposite spin directions are superposed. The wave function in the beam I_H behind the interferometer can be written as

$$|\psi\rangle = \frac{1}{\sqrt{2}}\left(|\tfrac{1}{2}\rangle + e^{i\delta}|-\tfrac{1}{2}\rangle\right) \tag{15}$$

where δ is a phase shift produced in P (Figure 1). In turn, the analyzer is such that it can detect the spin in any direction. This can be achieved by an accelerator coil which produces a variable magnetic field. The Larmor frequency within the coil can be varied by changing the current. Thus, the polarization vector can be turned by an additional angle φ. As a result, it can be made to assume an arbitrary direction in the x–y plane. Next, the Heusler single-crystal analyzer reflects only that part of the intensity which corresponds to a polarization parallel to the $+x$ axis. The intensity of the forward beam I_H has been registered as a function of δ (this angle can be changed in the neutron interferometer by means of phase shifter P) and the angle φ produced by the accelerator coil. The experiment was performed to establish $I = I(\delta)$ in the $+x$ and the $+z$ directions. Only in the $+x$ direction has the interference pattern been obtained.

According to the authors of this experiment it shows that the superposition of opposite spin eigenstates is a new state with the polarization vector perpendicular to both of the spin states existing before superposition.

We can look at this experiment a little bit differently. The neutron interferometer can be treated only as a preparation device. The neutron interferometry gives in principle the possibility of generating every state for spin 1/2. In this case, every state can be represented as

$$|\psi\rangle = |\mathbf{n}\rangle \to |\varphi, \theta\rangle = \cos\frac{\theta}{2}\, e^{i\varphi/2}|\tfrac{1}{2}\rangle + \sin\frac{\theta}{2}\, e^{-i\varphi/2}|-\tfrac{1}{2}\rangle \tag{16}$$

up to the phase β, $|\beta|=1$. Experiments have been reported (Zeilinger, 1979) in which every state (16) could be prepared. A partial absorber (magnetic phase shifter) was placed in one beam path inside the interferometer. Behind the interferometer we have

$$|\varphi\rangle=(1+e^{-\alpha})^{-1/2}(|\tfrac{1}{2}\rangle+e^{i\delta}\,e^{-\alpha/2}|-\tfrac{1}{2}\rangle) \tag{17}$$

where $e^{-\alpha/2}$ describes the absorber. For $e^{-\alpha/2}=\operatorname{tg}(\theta/2)$ we have (16).

Now, for one established δ our preparation device produces the state (15). The analyzer measures the intensity only as a function of φ. The Heusler crystal reflects neutrons in the $+x$ direction and can be described by the projector $P_x=|\psi_x\rangle\langle\psi_x|$, where $|\psi_x\rangle=(2)^{-1/2}(|\frac{1}{2}\rangle+|-\frac{1}{2}\rangle)$. The action of the accelerator coil could be expressed by the unitary operator $U_\varphi=\exp(-i\varphi\mathcal{J}_3)$.

We have $I\sim|\langle\psi|U_\varphi\psi_x\rangle|^2$. Let us remark that $U_\varphi|\psi_x\rangle$ is equal to $|\omega(\varphi)\rangle$ up to the normalization constant $I\sim|\langle\psi|\omega(\varphi)\rangle|^2$. For the prepared state (15) we have

$$I\sim|\langle\psi|\omega(\varphi)\rangle|^2=1+\cos(\varphi+\delta) \tag{18}$$

Many years ago Carruthers and Nieto (1968) introduced the so-called cosine operator $C=\frac{1}{2}(P+P^*)$. Using the phase representation, we easily obtain for (15)

$$\langle\psi|C|\psi\rangle=\int_0^{2\pi}\cos\varphi|\psi(\varphi)|^2\,d\varphi|2\pi=\tfrac{1}{2}\cos\delta \tag{19}$$

We see that for $\varphi=0$

$$I(0)\sim\cos\delta\sim\langle\psi|C|\psi\rangle \tag{20}$$

Measuring $I=I(\varphi)$, we have the possibility of determining the angle δ, and $I(0)$ is proportional to the mean value of the cosine operator. Moreover, having δ, we can write

$$\int_\Delta I(\varphi)\,d\varphi/2\pi\sim(\psi,\,M(\Delta)\psi) \tag{21}$$

It allows us to determine the probability that the spin phase is in Δ when the initial state was given by (15). We can also apply the interpretation given by Schroeck (1986) to (21).

Two remarks seem to be indispensable. First, the proposal to measure $I=I(\varphi)$ and from that determine the angle δ is a purely theoretical one. I am not expert in the experimental area and am not prepared to evaluate presently existing technical possibilities and instrument sensitivity, but only the principle of such an experiment. This analysis is rather a proposal which should be investigated by experts in the field of neutron interferometry.

Second, it is tempting to treat generally the angle φ in the parametrization (16) as the value of the phase. In the case of spin 1/2, it is enough to take the projector $P_{\mathbf{n}} = |\mathbf{n}\rangle\langle\mathbf{n}|$ for $\theta = \pi/2$ to obtain $P(\varphi)$ from (14). The only difference is in the meaningless normalization factor. But this agreement is by chance. In fact, the description by means of $P_{\mathbf{n}}$ is rather connected with the direction of spin. The angles φ and θ from (16) have a very clear geometrical meaning. Every state $|\psi\rangle$ from $\mathscr{H} = \mathbb{C}^2$ could be represented as a unit vector in the sphere $S_2 = \{\mathbf{n} \in \mathbb{R}^3, \|\mathbf{n}\| = 1\}$ and φ and θ determine the direction of the vector $\mathbf{n}$. However, the equivalence between $P(\varphi)$ and $|\theta = \pi/2, \varphi\rangle\langle\theta = \pi/2, \varphi|$ is destroyed for $J > \frac{1}{2}$. To see this, we must analyze the description of the direction for spin within the extended quantum mechanical formalism with semiobservables and show what kind of connection there is between the phase and the direction of spin.

4. CONNECTION BETWEEN THE PHASE AND THE DIRECTION OF SPIN

There exists no state $|\varphi\rangle$ for which

$$\mathscr{J}|\varphi\rangle = \mathbf{n}|\varphi\rangle \tag{22}$$

where $\mathbf{n} = (\cos\varphi \sin\theta, \sin\varphi \sin\theta, \cos\theta)$. But the same question for the projection of $\mathscr{J}$ onto $\mathbf{n}$ is already reasonable and

$$(\mathscr{J} \cdot \mathbf{n})|\mathbf{n}, m\rangle = m|\mathbf{n}, m\rangle \tag{23}$$

For $m = J$ we have (Radcliffe, 1971; Perelomov, 1986)

$$|\mathbf{n}, J\rangle \equiv |\mathbf{n}\rangle = \sum_{m=-J}^{J} A_m(\theta)\, e^{-im\varphi}|m\rangle \tag{24}$$

where

$$A_m(\theta) = \binom{2J}{m}^{1/2} \left(\cos\frac{\theta}{2}\right)^{J+m} \left(\sin\frac{\theta}{2}\right)^{J-m} \tag{25}$$

Using (23), we can build the semiobservable

$$A(\Delta) = \int_{\Delta} |\mathbf{n}\rangle\langle\mathbf{n}|\, d\mathbf{n} \tag{26}$$

where $\Delta \in \mathscr{B}(S_2)$ is the Borel σ-algebra of S_2 and $d\mathbf{n} = [(2J+1)/4\pi] \sin\theta\, d\theta\, d\varphi$ is a measure on S_2. Owing to the overcompleteness property of (23) (see, for instance, Perelomov, 1986), we have $A(S_2) = I$. Expressions like (26) have often been discussed: for spin 1/2, by Schroeck (1982, 1985) and Busch (1986), and for arbitrary spin by Holevo (1982). In quantum mechanics we can speak about the direction of spin: it is the property connected with the semispectral measure (26). Let us remark that

$$\int_{S_2} \mathbf{n}|\mathbf{n}\rangle\langle\mathbf{n}|\, d\mathbf{n} = \frac{1}{J+1}\mathscr{J} \tag{27}$$

It is a spectral decomposition for $\mathscr{J}/J+1$ with respect to A. When $|\psi\rangle$ is an initial state, then $\int_\Delta |\langle\psi|\mathbf{n}\rangle|^2\, d\mathbf{n}$ is a probability of finding the direction of spin anywhere in Δ. For spin 1/2 we can use the formerly described apparatus to detect the intensity which corresponds to the above probability. We only must take into account the angle θ. It is the angle between the Oz and the direction of magnetic field in the Stern–Gerlach analyzer. If the prepared state is $|\psi\rangle = |\mathbf{n}'\rangle$, we will register the intensity proportional to $|\langle\mathbf{n}'|\mathbf{n}\rangle|^2 = (1+\mathbf{n}\cdot\mathbf{n}')/2$.

It turns out that the question about the direction of spin leads to the question about the phase.

First we see from (24) that $|\theta=\pi/2, \varphi\rangle\langle\theta=\pi/2)\varphi|$ is really different from $P(\varphi)$ in (14) for $J>1/2$.

The connection between (14) and (26) will become visible when we calculate the marginal density for semiobservable (26). We obtain

$$\begin{aligned} P(\theta) &= \int_0^{2\pi} |\mathbf{n}\rangle\langle\mathbf{n}|\, d\varphi/2\pi \\ &= \sum_{m=-J}^{J} A_m^2(\theta)|m\rangle\langle m| \end{aligned} \tag{28}$$

$$\begin{aligned} \tilde{P}(\varphi) &= \int_0^{\pi} |\mathbf{n}\rangle\langle\mathbf{n}| \frac{2J+1}{2}\sin\theta\, d\theta = \sum_{m,m'=-J}^{J} e^{i(m-m')\varphi} \\ &\times \frac{\Gamma(J+1-(m+m')/2)\Gamma(J+1+(m+m')/2)}{[(J+m)!(J-m)!(J+m')!(J-m')!]^{1/2}}|m\rangle\langle m'| \end{aligned} \tag{29}$$

where Γ is the Euler function. Let us introduce the following definition:

$$a_{mm'} = \frac{\Gamma(J+1-(m+m')/2)\Gamma(J+1+(m+m')/2)}{[(J+m)!(J-m)!(J+m')!(J-m')!]^{1/2}} \tag{30}$$

We remark that $0 < a_{mm'} \le (a_{mm}a_{mm'})^{1/2} = 1$ because $a_{mm} = 1$. When we compare (29) with $P(\varphi)$ from (14) we see that these two expressions differ only by the factors $a_{mm'}$.

In turn, $P(\theta)$ is connected with the $\mathscr{J}_3$. For fixed θ we have $\sum_m A_m^2(\theta) = 1$. The semispectral measure $P(\theta)$ for given θ looks like a density operator. The $A_m^2(\theta) = |\langle m|\mathbf{n}\rangle|^2$ are probabilities corresponding to all values of m in the detection by means of a Stern–Gerlach analyzer when the direction of this device (i.e., the magnetic field) is $\mathbf{n}$. This description of spin measurement contains information about the measurement device. In indirect form $P(\theta)$ carries properties of $\mathscr{J}_3$. For instance, we recognize the main result of the Stern–Gerlach experiment, i.e., $(2J+1)$ different deflections, because $P(\theta)|m\rangle = A_m^2(\theta)|m\rangle$. Moreover,

$$\int_0^\pi (J+1)\cos\theta\, P(\theta)\,\frac{2J+1}{2}\sin\theta\, d\theta = \sum_{m=-J}^{J} m|m\rangle\langle m| = \mathscr{J}_3 \tag{31}$$

Thus, we also have agreement on the level of the mean values.

For $A(\cdot)$ we can also restore the "phase-spin" uncertainty relation:

$$\Delta\mathscr{J}_3\,\Delta\varphi > \tfrac{1}{4} \tag{32}$$

where

$$\Delta\mathscr{J}_3 = \langle J_3^2\rangle - \langle\mathscr{J}_3\rangle^2,\ \langle\mathscr{J}_3\rangle = \langle\psi|\mathscr{J}_3|\psi\rangle,\ \Delta\varphi = |\langle e^{i\varphi}\rangle|^{-2} - 1, \langle e^{i\varphi}\rangle$$
$$= \int_0^{2\pi} e^{i\varphi}|\varphi(\varphi)|^2\, d\varphi|2\pi.$$

It is a somewhat modified version of the uncertainty relation for $\mathscr{J}_3$ and sine and cosine operators introduced by Carruthers and Nieto. For details see Carruthers and Nieto (1968) and Holevo (1982). Simple calculations show

$$\Delta_A\mathscr{J}_3 = \int_0^\pi (J+1)^2\cos^2\theta\,\langle\psi|P(\theta)|\psi\rangle\,\frac{2J+1}{2}\sin\theta\, d\theta - \langle\mathscr{J}_3\rangle^2 > \Delta\mathscr{J}_3 \tag{33}$$

$$\Delta_A\varphi = \left|\int_0^{2\pi}\frac{d\varphi}{2\pi}\, e^{i\varphi}\langle\psi|\tilde{P}(\varphi)|\psi\rangle\right|^{-2} - 1 > \Delta\varphi \tag{34}$$

Thus,

$$\Delta_A\mathscr{J}_3\,\Delta_A\varphi > \tfrac{1}{4} \tag{35}$$

The "phase-spin" uncertainty relation also has an entropic formulation. Using the same methods as for the pairs position–momentum, angle–angular momentum (Bialynicki-Birula and Mycielski, 1975), and phase–number of quanta (Grabowski, 1987), we have the inequality

$$-\int_0^{2\pi}\frac{d\varphi}{2\pi}|\psi(\varphi)|^2\ln|\psi(\varphi)|^2 - \sum_{m=-J}^{J}|c_m|^2\ln|c_m|^2 \geq 0 \tag{36}$$

The entropies in (36) are interpreted as the measures of uncertainty connected with the measurement of the phase for spin and the third component of spin. The sum of these entropies attains its lower bound if and only if the system is in an eigenstate $|m\rangle$. The inequality (36) is not trivial. The entropy $-\sum_m |c_m|^2\ln|c_m|^2 \geq 0$ because $|c_m|^2 \leq 1$. The same is not necessarily true for the first expression in (36). For instance, when $|\psi_x\rangle=(1/\sqrt{2})\times(|\frac{1}{2}\rangle+|-\frac{1}{2}\rangle)$

$$-\int_0^{2\pi}|\psi_x(\varphi)|^2\ln|\psi_x(\varphi)|^2\,d\varphi/2\pi=\ln 2-1<0 \tag{37}$$

For (26) there also exists a proposal for how to restrict the entropy from below. Several years ago Lieb (1978) conjectured the inequality

$$-\int_{S_2}|\langle\psi|\mathbf{n}\rangle|^2\ln|\langle\psi|\mathbf{n}\rangle|^2\,d\mathbf{n}\geq\frac{2J}{2J+1} \tag{38}$$

This conjecture is manifestly true for $J=\frac{1}{2}$ and the equality is attained for $|\psi\rangle=|\mathbf{n}'\rangle$. Up to now this conjecture has not been proved. Scutaru (1979) verified it positively for $|\psi\rangle=|m\rangle$. The question arises as to what kind of physical property describes (38).

I show that Lieb's conjecture reflects in a certain sense the "phase-spin" uncertainty relation.

Introduce the notations

$$f(\varphi)=|\psi(\varphi)|^2=\langle\psi|P(\varphi)|\psi\rangle=\sum_{m,m'=-J}^{J}\bar{c}_m c_{m'}\,e^{i(m-m')\varphi} \tag{39}$$

$$\tilde{f}(\varphi)=\langle\psi|\tilde{P}(\varphi)|\psi\rangle=\sum_{m,m'=-J}^{J}\bar{c}_m c_{m'}a_{mm'}\,e^{i(m-m')\varphi} \tag{40}$$

where $|\psi\rangle=\sum_m c_m|m\rangle$ and $S(f)=-\int_0^{2\pi}f(\varphi)\ln f(\varphi)\,d\varphi/2\pi$.

I will show that

$$S(\tilde{f})\geq S(f) \tag{41}$$

The following relations are fulfilled:

$$\|\tilde{f}\|_1=\int_0^{2\pi}\frac{d\varphi}{2\pi}|\tilde{f}(\varphi)|=\sum_m|c_m|^2a_{mm}=\sum_m|c_m|^2=\int_0^{2\pi}\frac{d\varphi}{2\pi}|(\varphi)|=\|f\|_1 \tag{42}$$

$$\|\tilde{f}\|_\infty=\sup|\tilde{f}|\leq\sum_{m,m'}|c_m||c_{m'}|a_{mm'}\leq\left(\sum_m|c_m|\right)^2=\|f\|_\infty \tag{43}$$

because $a_{mm}=1$ and $a_{mm'}\leq 1$. Using the Riesz–Thorin theorem (Reed and Simon, 1975), we have

$$\|\tilde{f}\|_p\leq\|f\|_p \tag{44}$$

where $\|f\|_p=[\int_0^{2\pi}|f(\varphi)|^p\,d\varphi/2\pi]^{1/p}$ and $1\leq p\leq\infty$. Taking the derivation of (44) at $p=1$, we obtain (41). By means of a similar technique, we can prove

$$S(\langle\psi|P(\theta)|\psi\rangle)\geq-\sum_m|c_m|^2\ln|c_m|^2 \tag{45}$$

Now, from the subadditivity of entropy we have

$$S(\langle\psi|P(\theta)\psi\rangle)+S(\langle\psi|\tilde{P}(\varphi)|\psi\rangle)\geq S(|\langle\mathbf{n}|\psi\rangle|^2)\geq\frac{2J}{2J+1} \tag{46}$$

Taking into account (41), (45), and (36), finally, we have that

$$S(\langle\psi|P(\theta)|\psi\rangle)+S(\langle\psi|\tilde{P}(\varphi)|\varphi\rangle)\geq 0 \tag{47}$$

Comparison of the two last inequalities suggests that Lieb's conjecture could carry information about the "phase-spin" uncertainty relation.

We see that the description of the direction of spin is closely related to the spin phase.

5. DISCUSSION

Within the probabilistic scheme of quantum mechanics, I have provided the description for the spin phase. It is connected with the family of states $|\omega(\varphi)\rangle=\sum_m e^{im\varphi}|m\rangle$. These states determine the spin phase operator as a semispectral measure. Owing to neutron interferometry, there is the possibility of preparing and detecting the states $|\omega(\varphi)\rangle$ for spin one-half. To describe the results of these experiments the notion of the polarization vector of spin, i.e., $\langle\mathbf{n}|\mathcal{J}|\mathbf{n}\rangle=J\mathbf{n}$ was used. In such a description the difference between states and observables disappears. Instead of any observable in a state, we may think in terms of the classical vector $\mathbf{n}$. For spin 1/2 this is possible because every state can be represented as $|\mathbf{n}\rangle$ and such a classical picture is justified. For $J>\frac{1}{2}$ we must use a more sophisticated formalism such as presented above, because not every state has a parametrization by $\mathbf{n}$. In other words, the P representation for density operators $[\rho=\int_{S_2}P(\mathbf{n})|\mathbf{n}\rangle\hat{\lambda}$ $\langle\mathbf{n}|\,d\mathbf{n}]$ for spin one-half always has $P(\mathbf{n})\geq 0$ and we can look at spin states very much in a classical sense. But for higher spins there exist states with $P(\mathbf{n})\leq 0$ and they should show highly nonclassical properties similar to the "antibunching" and "squeezed" states of harmonic oscillators. Experimental investigations of such states should be very interesting.

I have discussed the possibility of describing the direction of spin in quantum mechanics. Conceptually, the spin phase and the direction of spin are two different notions. Nevertheless, the $A(\cdot)$ carries some knowledge about the phase. These two levels of description are not the same even for $J=\frac{1}{2}$. Taking (14) and (29), we have

$$P(\varphi)=e^{i\varphi}|\tfrac{1}{2}\rangle\langle-\tfrac{1}{2}|+e^{-i\varphi}|-\tfrac{1}{2}\rangle\langle\tfrac{1}{2}|+I \tag{48}$$

$$P(\varphi)=\frac{\pi}{4}(e^{i\varphi}|\tfrac{1}{2}\rangle\langle-\tfrac{1}{2}|+e^{-i\varphi}|-\tfrac{1}{2}\rangle\langle\tfrac{1}{2}|)+I \tag{49}$$

In the language of intensities, this means

$$I(\varphi)=\frac{\pi}{4}\,I(\varphi)+\tfrac{3}{4}\pi \tag{50}$$

The question arises: What kind of description, (48) or (49), should be adopted to explain the experiment in Figure 1. What kind of intensity is measured by the experimental device $\tilde{I}$ or I? Is this device sensitive enough to distinguish between the two approaches?

To conclude, I remark that this presentation of the connection between A and $\mathscr{J}_3$, M is very much in the spirit of the description for the simultaneous measurement of position and momentum (Davies, 1976; Busch, 1985; Prugovecki, 1984).

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